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The Nowicki conjecture for the free metabelian Leibniz algebra of rank 2

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***Abstract*—Let *L* be the free metabelian right Leibniz algebra generated by *x* and *K* over a field *K* of characteristic zero, and let *K*[*u*,*v*] be the polynomial algebra generated by commuting variables *u*=*adx* and *v*=*ady* acting on the commutator ideal [*L*,*L*] of the Leibniz algebra *L* as adjoint operators. Let *d* be the linear nilpotent derivation of *L* such that d(y)=x, d(x)=0. In this study, we give free generators of the algebra [*L*,*L*]^d of constants consisting of elements sent to zero by d, as a *K*[*u*,*v*]^d -module.**

Keywords—algebra of constants; right Leibniz algebra; the Nowicki conjecture

I. Introduction

Consider the polynomial algebra $K[x_1,y_1,\dots,x_n,y_n]$ of rank $2n$ over a field K of characteristic zero. Let δ be the linear nilpotent derivation such that $\delta(y_i)=x_i$, $\delta(x_i)=0$ for each $i=1,\dots,n$. The algebra

$$K[x_1,y_1,\dots,x_n,y_n]^\delta=\{a:\delta(a)=0\}$$

of constants of δ is known to be finitely generated by Weitzenböck [1]. The generators were conjectured by Nowicki [2] as

$$x_1,\dots,x_n,x_iy_j-y_jx_i$$

where $1\le i<j\le n$. This was proved by many mathematicians, see e.g. [3,4,5]. For the nonassociative or noncommutative settings of the problem, see [6,7].

The concept of Leibniz algebras appeared first in the works [8,9] of Loday in 1993. Let L be the free metabelian Leibniz algebra of rank 2. In this study, we study the Nowicki conjecture in the Leibniz algebra L, and give generators of the algebra of constants.

II. Preliminaries

Let K be a field of characteristic zero. A vector space $\boldsymbol{V}$ is a Leibniz algebra with the bilinear commutator $[.,.]:V\to V$ satisfying the identity

$$\big[[x,y],z\big]=\big[[x,z],y\big]+\big[x,[y,z]\big].$$

One may deduce from the definition that Leibniz algebras are not associative. In particular, adding the skew symmetry to the definition, we obtain a Lie algebra. A Leibniz algebra satisfying the metabelian identity

$$\big[[x,y],[z,t]\big]=0$$

is called metabelian. Let L be the free metabelian Leibniz algebra of rank 2 generated by x and y over the base field K. The metabelian identity implies that the commutator ideal $L'=[L,L]$ of the Leibniz algebra L is a right $K[u,v]$-module by the action

$$[x,y]u=\big[[x,y],x\big]$$

$$[x,y]v=\big[[x,y],y\big]$$

Drensky and Cattanneo [10] showed that L is of a basis consisting of x , y and elements of the form

$$[x,y]u^kv^l=\Big[\dots\Big[\big[\dots\big[[x,y],x\big],\dots,x\big],y\Big],\dots,y\Big]$$

where $0\le k,l$. It is also well known that L' is a free right $K[u,v]$-module generated by $[x,x]$, $[x,y]$, $[y,x]$, $[y,y]$ (see e.g. [11])

Let δ be the linear derivation of L such that $\delta(y)=x$, $\delta(x)=0$. The algebra

$$L^\delta=\{p\in L:\delta(p)=0\}$$

is called the algebra of constants of δ. Consider the derivation acting on the polynomial algebra by the rule

$$\delta(v)=u,\ \delta(u)=0$$

Then direct computations give that $K[u,v]^\delta=K[u]$. This is also a particular case of the Nowicki conjecture stated in [2]. The structure of the commutator ideal L' yields that $(L')^\delta$ is a right $K[u,v]^\delta$-module. We close this section by the next technical lemma, whose proof is straightforward, and that will be vital in the proof of the main theorem.

Lemma 1: For $0\le n$, the vector space

$$K[u,v]^{\delta^{n+1}}=\{a\in K[u,v]:\delta^{n+1}(a)=0\}$$

is equal to the direct sum

$$K[u]\oplus yK[u]\oplus\cdots\oplus y^nK[u].$$

III. Main Results

In the next theorem, we present out main result.

Theorem 2: The right $K[u,v]^\delta$-module $(L')^\delta$ is freely generated by

$$[x,x],\ \ [x,y]-[y,x],\ \ [x,x]v-[y,x]u$$
$$[x,x]v^2-[x,y]uv-[y,x]uv+[y,y]u^2$$

Proof. Clearly, the elements in the statement of the theorem are δ-constants. Now let

$$p=[x,x]w_1+[x,y]w_2+[y,x]w_3+[y,y]w_4\in L'$$

for some $w_1,w_2,w_3,w_4\in K[u,v]$. Assuming that $p\in(L')^\delta$ we obtain that

$$0=[x,x]\delta(w_1)+[x,x]w_2+[x,y]\delta(w_2)+[x,x]w_3$$
$$+[y,x]\delta(w_3)+[x,y]w_4+[y,x]w_4+[y,y]\delta(w_4)$$

This yields that

(1) $\delta(w_1)+w_2+w_3=0$

(2) $\delta(w_2)+w_4=0$

(3) $\delta(w_3)+w_4=0$

(4) $\delta(w_4)=0$

in the $K[u,v]$-module L' freely generated by $[x,x]$, $[x,y]$, $[y,x]$, $[y,y]$. Hence,

$$\delta^3(w_1)=\delta^2(w_2)=\delta^2(w_3)=\delta(w_4)=0.$$

Therefore,

$$w_1=a_0+a_1v+a_2v^2$$
$$w_2=b_0+b_1v$$
$$w_3=c_0+c_1v$$
$$w_4=d_0$$

for some $a_0,a_1,a_2,b_0,b_1,c_0,c_1,d_0\in K[u,v]^\delta=K[u]$ by by Lemma 1. Substituting these new expressions in equations (2) and (3) we have that

$$b_1u+d_0=c_1u+d_0=0$$

implying that

$$d_0=b_1u\ ,\ b_1=c_1.$$

Now the equation (1) yields that

$$a_1u+2a_2uv+2b_1v\ +b_0+c_0=0$$

which implies

$$a_1u+b_0+c_0=0$$
$$2a_2u+2b_1\ =0$$

or

$$b_1=-a_2u$$
$$c_0=-b_0-a_1u$$

and $d_0=-a_2u^2$. As a result

$$w_1=a_0+a_1v+a_2v^2$$
$$w_2=b_0-a_2uv$$
$$w_3=-b_0-a_1u-a_2uv$$
$$w_4=-a_2u^2$$

which proves that the right $K[u]$-module $(L')^\delta$ is generated by

$$p_1=[x,x],\ \ p_2=[x,y]-[y,x],\ \ p_3=[x,x]v-[y,x]u$$
$$p_4=[x,x]v^2-[x,y]uv-[y,x]uv+[y,y]u^2.$$

The rest is to show that p_1,p_2,p_3,p_4 are free generators. Let

$$p_1a+p_2b+p_3c+p_4d=0$$

for some $a,b,c,d\in K[u]$. Then

(5) $a+bv+cv^2=0$

(6) $d-cuv=0$

(7) $d+bu+cuv=0$

(8) $cu^2=0$

in the $K[u,v]$-module L' freely generated by $[x,x]$, $[x,y]$, $[y,x]$, $[y,y]$. Initially, the equation (8) gives $c=0$ in the integral domain $K[u]$. Next, (6) implies $d=0$ substituting the fact $c=0$ in it. Now (7) turns into $bu=0$, and similarly $b=0$. Finally $a=0$ by (1). This completes the proof.

Corollary 3: L^δ has the following structure.

$$L^\delta=Kx\oplus p_1K[u]\oplus p_2K[u]\oplus p_3K[u]\oplus p_4K[u].$$

IV. Conclusion

This study provides description of the algebra of constants of the free metabelian right Leibniz algebra of rank 2. One may generalize the result for greater ranks.

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