# Applied Residue-Based Seeded Secant Method For Computing Eccentricity of Normal Ellipsoid

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***Abstract*— In this paper, applied residue-based seeded secant method for computing eccentricity of normal ellipsoid is presented. A residue-based seeded secant algorithm is a form of secant algorithm that requires only one initial guess value from the user and then generates the second initial guess root from the single guess root. In this paper, a residue-based seeded secant algorithm was adapted for the computation of the eccentricity of the normal ellipsoid. A set of normal ellipsoid constant data were used for a numerical example. The results showed that the residue-based seeded secant algorithm obtained the eccentricity in two iterations with error tolerance of the order of $x\,10^{-5}$. Essentially, the residue-based seeded secant method converges fast when applied to the computation of the eccentricity**.

***Keywords*— Seeded Secant, Eccentricity, Iteration Algorithm, Normal Ellipsoid , Residue-Based Seeded Secant**

## 1. INTRODUCTION

Generally, numerical iteration methods are used to solve complex equations, especially those that do not have closed-form solutions [1,2,3,4,5,6,7,8]. Among the numerous iteration approaches, secant iteration approach has become one of the popular methods because of its fast converge and ease of implementation [9,10,11,12,13]. Unlike the Regular Falsi and bisection methods, the secant method does not require that the initial two guess roots should bracket the actual root [14,15,16,17,18]. As such, it is easier to select the two initial guess roots for the secant method.

In addition, the secant method is similar to the popular Newton Raphson method. However, the secant method is preferred to the Newton Raphson method because the secant method does not require the computation of the first derivative of the function [19,20,21].

Furthermore, in order to simplify the application of the secant method, a modified seeded version of the secant algorithm which requires only one initial guess root value has been developed. Particularly, in this paper, a residue-based seeded secant method for computing the eccentricity of normal ellipsoid is presented. Specifically, a single initial guess value of the square of the eccentricity ($e^2$) of normal ellipsoid is presented and then used to compute the complementary value of the $e^2$. The difference between the initial guess value of $e^2$ and the computed complementary value of $e^2$ is the residue which is then used to modify the next guess root values that are employed in the secant algorithm to iteratively determine the actual value of e. The detailed mathematical analysis and algorithm of the iteration scheme are presented and implemented in Matlab software.

## 2. METHODOLOGY

### 2.1 Expression for Computing Eccentricity of Normal Ellipsoid

The normal ellipsoid first eccentricity, e is given as [22];

$$e^2 = 3J_2 + \frac{\omega^2(a^3)}{GM}\left[\left(\frac{4}{15}\right)\left(\frac{e^3}{2q_0}\right)\right] \quad (1)$$

$$e' = \frac{e}{\sqrt{1-e^2}} \quad (2)$$

$$q_0 = \frac{1}{2}\left[\left(1+\frac{3}{e'^2}\right)tan^{-1}(e') - \frac{3}{e'}\right] \quad (3)$$

Where

$\omega$ is the angular velocity ( $\omega = 7.292115 \text{ x } 10^{-5}$)
a is the equatorial earth radius (a = 6378137 km),
$J_2$ is the earth's dynamic form factor ($J_2 = 1.08263 \text{ x } 10^{-3}$)
GM is the geocentric gravitational constant (GM = $3.986005 \text{ x } 10^{14}\ m^3/s^2$,

The initial values of $e^2$ denoted as $e_0^2$ is given in [22] as ;

$$e_0^2 = 3J_2 + \frac{\omega^2(a^3)}{GM} \quad (4)$$

The expression for $e^2$ in Eq 1 does not have a closed-form solution. Rather iterative solution approaches are used. In this paper, residue-based seeded secant iteration is used. In this case, the residue or error, denoted as $f(e_0^2)$ for $e_0^2$ or $f(e^2)$ for $e^2$ is computed and used to determine when the solution with the desired error tolerance is achieved. Specifically, for $e^2$ the residue, $f(e^2)$ is given as;

$$f(e^2) = e^2 - \left\{3J_2 + \frac{\omega^2(a^3)}{GM}\left[\left(\frac{4}{15}\right)\left(\frac{e^3}{2q_0}\right)\right]\right\} \quad (5)$$

## 2.2 The Residue-Based Seeded Secant Algorithm For Computing Eccentricity of Normal Ellipsoid

The Residue-Based Seeded Secant (RBSS) iteration uses a single initial value to carry out the iteration process. In the computation of $e^2$, the RBSS iteration starts with $e_0^2$ as given in Eq 4. Then, the second initial value denoted as $e_1^2$ is computed where ;

$$e_1^2 = e_0^2 + f(e_0^2) = e_0^2 + \left[e_0^2 - \left\{3J_2 + \frac{\omega^2(a^3)}{GM}\left[\left(\frac{4}{15}\right)\left(\frac{e_0^3}{2q_0}\right)\right]\right\}\right] \quad (6)$$

Which gives;

$$e_1^2 = 2e_0^2 - \left\{3J_2 + \frac{\omega^2(a^3)}{GM}\left[\left(\frac{4}{15}\right)\left(\frac{e_0^3}{2q_0}\right)\right]\right\} \quad (7)$$

Where;

$$e' = \frac{\sqrt{e_0^2}}{\sqrt{1-e_0^2}} \quad (8)$$

$$q_0 = \frac{1}{2}\left[\left(1+\frac{3}{e_0^2}\right)tan^{-1}\left(\sqrt{e_0^2}\right) - \frac{3}{\sqrt{e_0^2}}\right] \quad (9)$$

Next, $f(e_1^2)$ is computed and the classical secant formula for computing the expected root, $e^2$ from the two initial guess roots, $e_0^2$ and $e_1^2$ is applied to compute $e^2$ where;

$$e^2 = e_1^2 - \left\{f(e_1^2)\left(\frac{e_1^2 - e_0^2}{f(e_1^2) - f(e_0^2)}\right)\right\} \quad (10)$$

The , $f(e^2)$ is computed and compare with the error tolerance, ε. If $f(e^2) < \varepsilon$ then $e^2$ is the solution otherwise, $e_0^2 = e^2$ and then $e_1^2$ is recomputed in terms of $e_0^2$ and also $e^2$ is recomputed in terms of $e_0^2$ and $e_1^2$. The complete algorithm for the Residue-Based Seeded Secant (RBSS) iteration is presented as follows;

**Step 1.:** Input the values of the normal ellipsoid geocentric constants and computation error tolerance

**Step 1.1:** Input a = 6378137 km)

**Step 1.2:** GM= $3.986005\,x\,10^{14}$ $m^3/s^2$,

**Step 1.3:** $J_2 = 1.08263\,x\,10^{-3}$

**Step 1.4:** $\omega = 7.292115\,x\,10^{-5}$.

**Step 1.5:** Error Tolerance, $\varepsilon = 10^{-15}$

**Step 2.:** compute the initial value , $e_0^2$ of the eccentricity of the normal ellipsoid

$$e_0^2 = \frac{3J_2 + \frac{\omega^2(a^3)}{GM}}{\left(1+\frac{\omega^2(a^3)}{GM}\left(\frac{9}{14}\right)\right)}$$

**Step 3.:** compute the error, $f(e_0^2)$ in the values of $e_0^2$ where

**Step 3.1:** compute the second eccentricity, $e_0'$ in terms of $e_0^2$ where

$$e_0' = \frac{\sqrt{e_0^2}}{\sqrt{1-e_0^2}}$$

**Step 3.1:** compute $q_0$ in terms of $e_0^2$ where;

$$q_0 = \frac{1}{2}\left[\left(1+\frac{3}{e_0'^2}\right)tan^{-1}(e_0') - \frac{3}{e_0'}\right]$$

**Step 3.2:** compute the error, $f(e_0^2)$ in the values of $e_0^2$ where

$$f(e_0^2) = e_0^2 - \left\{3J_2 + \frac{\omega^2(a^3)}{GM}\left[\left(\frac{4}{15}\right)\left(\frac{e_0^3}{2q_0}\right)\right]\right\}$$

**Step 4.:** compute the next value , $e_1^2$ of the eccentricity of the normal ellipsoid, where

$$e_1^2 = e_0^2 + f(e_0^2)$$

**Step 5.:** compute the error, $f(e_1^2)$ in the values of $e_1^2$ where

**Step 5.1:** compute the second eccentricity, $e_1'$ in terms of $e_1^2$ where

$$e_1' = \frac{\sqrt{e_1^2}}{\sqrt{1-e_1^2}}$$

**Step 5.1:** compute $q_0$ in terms of $e_1^2$ where;

$$q_0 = \frac{1}{2}\left[\left(1+\frac{3}{e_1'^2}\right)tan^{-1}(e_1') - \frac{3}{e_1'}\right]$$

**Step 5.2:** compute the error, $f(e_1^2)$ in the values of $e_0^2$ where

$$f(e_1^2) = e_1^2 - \left\{3J_2 + \frac{\omega^2(a^3)}{GM}\left[\left(\frac{4}{15}\right)\left(\frac{e_1^3}{2q_0}\right)\right]\right\}$$

**Step 6.:** compute the expected value , $e^2$ of the eccentricity of the normal ellipsoid, where

$$e^2 = e_1^2 - \left\{f(e_1^2)\left(\frac{e_1^2 - e_0^2}{f(e_1^2) - f(e_0^2)}\right)\right\}$$

**Step 7.:** compute the error, $f(e^2)$ in the values of $e^2$ where

**Step 7.1:** compute the second eccentricity, $e'$ in terms of $\mathbf{e'^2}$ where

$$e' = \frac{\sqrt{e^2}}{\sqrt{1-e^2}}$$

**Step 7.1:** compute $q_0$ in terms of $e^2$ where;

$$q_0 = \frac{1}{2}\left[\left(1+\frac{3}{\mathbf{e'^2}}\right)tan^{-1}(e') - \frac{3}{e'}\right]$$

**Step 7.2:** compute the error, $f(e^2)$ in the values of $e^2$ where

$$f(e^2) = e^2 - \left\{3J_2 + \frac{\omega^2(a^3)}{GM}\left[\left(\frac{4}{15}\right)\left(\frac{e^3}{2q_0}\right)\right]\right\}$$

**Step 8.:** Check the tolerance error

**Step 8.1:** If $f(e^2) > \varepsilon$ Then

**Step 8.2:** $e_0^2 = e^2$

**Step 8.3:** GOTO Step 3

**Step 8.4:** ELSE

**Step 8.5:** Print "Square of First eccentricity is $e^2$ =", $\sqrt{e^2}$

**Step 8.6:** Print "First eccentricity is e =', $\sqrt{e^2}$

**Step 8.7:** Print "Second eccentricity is $e'$ =", $e'$

**Step 8.6:** EndIf

**Step 9.:** Stop

## 3 RESULTS AND DISCUSSION

A numerical example was conducted using the data values provided in the algorithm. The initial value of the eccentricity and other parameters were obtained and given as follows;

i) $e_0^2$ =0.006709281393112260

ii) $e'$ = 0.082186372985854000

iii) $q_0$ = 0.000073592001058387

iv) $f(e_0^2)$ =-1.49345515727860 $x\ 10^{-5}$

Then, the second initial value, $e_1^2$ is obtained as $e_1^2 = e_0^2 + f(e_0^2)$ . The RBSS algorithm used the values of $e_0^2$ and $e_1^2$ iteratively to determine the value of $e^2$ with error tolerance, $\varepsilon = 10^{-15}$. The results are shown in Table 1. The results show that the initial of $e_0^2$ gave error tolerance, $\varepsilon = 10^{-5}$ but after two iterations the Residue-Based Seeded Secant (RBSS) algorithm gave $e^2 = 0.006694380023$ with error tolerance, $\varepsilon = -7.701305E - 15$.

Table 1 The results of the Residue-Based Seeded Secant (RBSS) iteration for computing the first eccentricity of normal ellipsoid

| Cycle | $e_0^2$ | $f(e_0^2)$ | $e_1^2$ | $f(e_1^2)$ | $e^2$ | $f(e^2)$ |
|---|---|---|---|---|---|---|
| 0 | 0.006709281393 | -1.493455E-05 | 0.006694346842 | 3.325510E-08 | 0.006694380023 | 3.476559E-14 |
| 1 | 0.006694380023 | 3.476559E-14 | 0.006694380023 | 1.937877E-13 | 0.006694380023 | 2.034354E-13 |
| 2 | 0.006694380023 | 2.034354E-13 | 0.006694380023 | -9.835882E-15 | 0.006694380023 | -7.701305E-15 |
| 3 | 0.006694380023 | -7.701305E-15 | 0.006694380023 | -5.946632E-15 | 0.006694380023 | 0.000000E+00 |
| 4 | 0.006694380023 | 0.000000E+00 | 0.006694380023 | 0.000000E+00 | | |

## 4. Conclusion

**A** Residue-Based Seeded Secant (RBSS) algorithm was presented for computing the eccentricity of normal ellipsoid. A same normal ellipsoid constants dataset were used to demonstrate the application of the RBSS algorithm. The results showed that the RBSS algorithm obtained the eccentricity in two iterations with error tolerance of the order of $x\ 10^{-5}$.

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