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# A Non-Iterative Solution For The Eccentricity Of The Normal Ellipsoid

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***Abstract*— A non-iterative solution for the first eccentricity (e) of the normal ellipsoid is presented. The solution is based on a series approximation of $\tan^{-1}(x)$ which yielded a series approximation of the first eccentricity, e. The first five terms of the series was then considered in the non-iterative solution approach and numerical example was used to demonstrate the application of the solution approach. At the same time, a bisection iteration approach was also used to determine the value of the first eccentricity, e and the result was compared with that obtained from the non-iterative solution approach. The results show that it took about 10 iterations for the bisection method to arrive at the solution based on the initial value, $e_0^2 = 0.006709281393112260$. The error between the initial value $e_0^2 = 0.006709281393112260$ and the actual value of $e^2 = 0.006694387918274080$ was $1.489347483820000000 \times 10^{-5}$. On the other hand, the non-iterative solution approach presented in this paper does not require any iteration to obtain the value of first eccentricity (e) with the desired tolerance error, $\varepsilon$ in the order of $10^{-9}$.**

***Keywords*— *Normal Ellipsoid, First Eccentricity, Geocentric , Second Eccentricity, Equipontential Ellipsoid, Geocentric Gravitational Constant***

## I. INTRODUCTION

In the geodetic reference system, the earth is modeled as geocentric equipontential ellipsoid of revolution (or normal ellipsoid) [1,2,3,4,5,6] characterized by the following basic or primary constants; equatorial earth radius (a = 6378137 km), the geocentric gravitational constant for both the earth and the atmosphere (GM = $3.986005 \times 10^{14}$ $m^3/s^2$, the earth's dynamic form factor ($J_2 = 1.08263 \times 10^{-3}$) and angular velocity ($\omega = 7.292115 \times 10^{-5}$. In addition to these basic constants, the square of the first eccentricity is a fundamental [6,7,8,9,10,11,12] parameter which is derived from the four listed basic constants.

Notably, the first eccentricity is essential for the determination of some other derived parameters of the geocentric equipontential ellipsoid [5,13,14]. However, the square of the first eccentricity ($e^2$) is related to the basic constants via a transcendental equation which requires iterative method for its solution. However, in this paper, a non-iterative solution is derived for computing the first eccentricity of the normal ellipsoid. The approach in this paper utilized a series expansion of key component of the transcendental equation relating the first eccentricity to the basic or primary constants. The details of the derivation of the non-iterative solution are presented along with numerical example and estimation error performance analysis.

## II. METHODOLOGY

### A. *Analytical Expression for First Eccentricity of Normal Ellipsoid*

The first eccentricity, e of the normal ellipsoid is defined in terms of certain constants among which are the equatorial earth radius (a = 6378137 km), the geocentric gravitational constant for both the earth and the atmosphere (GM = $3.986005 \times 10^{14}$ $m^3/s^2$, the earth's dynamic form factor ( $J_2 = 1.08263 \times 10^{-3}$) and angular velocity ( $\omega = 7.292115 \times 10^{-5}$. The eccentricity, e is defined as follows [5];

$$e^2 = 3J_2 + \frac{\omega^2(a^3)}{GM}\left[\left(\frac{4}{15}\right)\left(\frac{e^3}{2q_0}\right)\right] \quad (1)$$

$$q_0 = \frac{1}{2}\left[\left(1 + \frac{3}{e'^2}\right)tan^{-1}(e') - \frac{3}{e'}\right] \quad (2)$$

$$e' = \frac{e}{\sqrt{1-e^2}} \quad (3)$$

### B. *Non-Iterative Solution to the Analytical Expression for First Eccentricity of Normal Ellipsoid*

In a bid to provide initial value for an iterative solution to e, the author in [5] used a series expansion of $\tan^{-1}(x)$ to provide an approximation of e as follows;

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots, \text{ for } |x| < 1 \quad (4)$$

Hence,

$$q_0 = \frac{1}{2}\left[\left(1+\frac{3}{e'^2}\right)\left\{e' - \frac{1}{3}e'^3 + \frac{1}{5}e'^5 - \frac{1}{7}e'^7 + \cdots\right\} - \frac{3}{e'}\right], \text{ for } |x| < 1 \quad (5)$$

Further expansion and simplifications were performed in [5] led to the following expression;

$$\left(\frac{4}{15}\right)\left(\frac{e^3}{2q_0}\right) = 1 - \frac{9}{14}e^2 - \frac{13}{392}e^4 - \frac{4189}{181104}e^6 + \frac{1720993}{3380608}e^8 + \cdots \quad (6)$$

Accordingly, [5] approximated $\left(\frac{4}{15}\right)\left(\frac{e^3}{2q_0}\right) = 1$ thereby making the initial value, $e_0^2$ of $e^2$ in Eq 1 as;

$$e_0^2 = 3J_2 + \frac{\omega^2(a^3)}{GM} \quad (7)$$

The approach in this paper utilizes the first 5 terms (in Eq 6) of the series for $\left(\frac{4}{15}\right)\left(\frac{e^3}{2q_0}\right)$. First, the first two terms in Eq 6 were used to determine $e_0^2$ and them, the 3rd, 4th and 5th terms were further computed using $e_0^2$ to obtain the effective value, $e_{0e}^2$ of $e^2$ which is more accurate and may not require further iteration for acceptable level of accuracy. In this case, $\left(\frac{4}{15}\right)\left(\frac{e^3}{2q_0}\right)$ is first approximated to $1 - \frac{9}{14}e^2$ which gives;

$$e_0^2 = 3J_2 + \frac{\omega^2(a^3)}{GM}\left(1 - \frac{9}{14}e_0^2\right) = 3J_2 + \frac{\omega^2(a^3)}{GM} - \frac{\omega^2(a^3)}{GM}\left(\frac{9}{14}e_0^2\right) \quad (8)$$

Hence,

$$e_0^2 = \frac{3J_2 + \frac{\omega^2(a^3)}{GM}}{\left(1 + \frac{\omega^2(a^3)}{GM}\left(\frac{9}{14}\right)\right)} \quad (9)$$

Recall that the first 5 terms in the series approximation of $\left(\frac{4}{15}\right)\left(\frac{e^3}{2q_0}\right)$ includes $1 - \frac{9}{14}e^2 - \frac{13}{392}e^4 - \frac{4189}{181104}e^6 + \frac{1720993}{3380608}e^8$. In that case, when all the five terms are considered, $e_0^2$ becomes;

$$e_0^2 = 3J_2 + \frac{\omega^2(a^3)}{GM} - \frac{\omega^2(a^3)}{GM}\left(\frac{9}{14}e_0^2\right) - \left(\frac{\omega^2(a^3)}{GM}\right)\left(\frac{13}{392}e^4\right) - \left(\frac{\omega^2(a^3)}{GM}\right)\left(\frac{4189}{181104}e^6\right) + \left(\frac{\omega^2(a^3)}{GM}\right)\left(\frac{1720993}{3380608}e^8\right) \quad (10)$$

Since the first two terms $1 - \frac{9}{14}e^2$ had been used in the determination of $e_0^2 = \frac{3J_2 + \frac{\omega^2(a^3)}{GM}}{\left(1 + \frac{\omega^2(a^3)}{GM}\left(\frac{9}{14}\right)\right)}$, then, in terms of the remaining three terms, $e_0^2$ can be approximated as;

$$e_{0e}^2 = \frac{3J_2 + \frac{\omega^2(a^3)}{GM}}{\left(1 + \frac{\omega^2(a^3)}{GM}\left(\frac{9}{14}\right)\right)} - \left(\frac{\omega^2(a^3)}{GM}\right)\left(\frac{13}{392}((e_0^2)^2)\right) - \left(\frac{\omega^2(a^3)}{GM}\right)\left(\frac{4189}{181104}(e_0^2)^3\right) + \left(\frac{\omega^2(a^3)}{GM}\right)\left(\frac{1720993}{3380608}(e_0^2)^4\right) \quad (11)$$

$$e^2 = e_{0e}^2 = e_0^2 - \left(\frac{\omega^2(a^3)}{GM}\right)\left(\frac{13}{392}((e_0^2)^2)\right) - \left(\frac{\omega^2(a^3)}{GM}\right)\left(\frac{4189}{181104}(e_0^2)^3\right) + \left(\frac{\omega^2(a^3)}{GM}\right)\left(\frac{1720993}{3380608}(e_0^2)^4\right) \quad (12)$$

## III. SIMULATION AND DISCUSSION OF RESULTS

The value of $e^2$ was iteratively computed using the initial value set by [5] Next, the value of $e^2$ was computed using the expression in Eq 11 (or Eq 12) derived in this paper. The results were compared. The input data for the simulation and computations are as follows;

i) a = 6378137 km)
ii) GM= $3.986005 x 10^{14}$ $m^3/s^2$,
iii) $J_2 = 1.08263 x 10^{-3}$
iv) $\omega = 7.292115 x 10^{-5}$.
v) Tolerance, $\varepsilon = 10^{-9}$

The iteration conducted in respect of $f(e^2)$ is such that the iteration started with the initial value, $e_0^2$ set by [5] where $e_0^2 = 3J_2 + \frac{\omega^2(a^3)}{GM}$. Hence;

$$f(e^2) = e^2 - \left\{3J_2 + \frac{\omega^2(a^3)}{GM}\left[\left(\frac{4}{15}\right)\left(\frac{e^3}{2q_0}\right)\right]\right\} \quad (13)$$

where $q_0$ is given in Eq 2 and $e'$ is given in Eq 3

Table 1 The results of the computation of $e^2$ using iterative approach based on the initial value defined by [5] and computation with the non-iterative approach presented in this paper

| Parameter Description | Result of iterative approach based on the initial value defined by [5] | Parameter Description | Result of the non-iterative approach presented in this paper |
|---|---|---|---|
| $e_{0e}^2 = e_0^2$ | 0.006709281393112260 | $e_{0e}^2$ | 0.006694390344701410 |
| $e_0$ | 0.081910203229587100 | $e_0$ | 0.08181925411968390 |
| $e'_0$ | 0.082186372985854000 | $e'_0$ | 0.08209450186751850 |
| $e^2$ | 0.006694387918274080 | $e^2$ | 0.006694387918274080 |
| $e^2 - e_0^2$ | -1.489347483820000000$x$ $10^{-5}$ | $e^2 - e_{0e}^2$ | -2.426427330760870000$x$ $10^{-9}$ |

Table 2 The results of the computation of $e^2$ using Bisection iterative approach based on the initial value defined by [5]

| Cycle | $e_L^2$ | $e_u^2$ | $f(e_L^2)$ (error based on $e_L^2$) |
|---|---|---|---|
| 1 | $e_0^2 = 0.006709281393112260$ | 0.006627287209056940 | -1.493455157278600000E-05 |
| 2 | 0.006709281393112260 | 0.006668284301084600 | -1.493455157278600000E-05 |
| 3 | 0.006709281393112260 | 0.006688782847098430 | -1.493455157278600000E-05 |
| 4 | 0.006699032120105340 | 0.006688782847098430 | -4.662456272660170000E-06 |
| 5 | 0.006699032120105340 | 0.006693907483601880 | -4.662456272660170000E-06 |
| 6 | 0.006696469801853610 | 0.006693907483601880 | -2.094432450029140000E-06 |
| 7 | 0.006695188642727750 | 0.006693907483601880 | -8.104205782618470000E-07 |
| 8 | 0.006694548063164820 | 0.006693907483601880 | -1.684145858188430000E-07 |
| 9 | 0.006694548063164820 | 0.006694227773383350 | -1.684145858188430000E-07 |
| 10 | 0.006694387918274080 | 0.006694227773383350 | -7.912831217517240000E-09 |
| 11 | $e^2 = 0.006694387918274080$ | 0.006694307845828720 | -7.912831217517240000E-09 |

The results of the computation of $e^2$ using the iterative approach based on the initial value defined by [5] and computation with the non-iterative approach presented in this paper are presented in Table 1. The solution value of e obtained with the non-iterative approach has tolerance error, $\varepsilon$ of $10^{-9}$. The results of the iterative approach show that it took about 10 iterations for the bisection method to arrive at the solution based on the initial value , $e_0^2 = 0.006709281393112260$ computed according to the formula presented in [5]. The error between the initial value $e_0^2 = 0.006709281393112260$ used in the iterative approach and the actual value of $e^2 = 0.006694387918274080$ is $1.489347483820000000\ x\ 10^{-5}$.

Notably, the approach presented in this paper does not require any iteration to achieve the desired tolerance error, $\varepsilon$ in the order of $10^{-9}$. However, it took about 10 iterations for the Bisection method to arrive at the solution with tolerance error of the order of $10^{-9}$.

## IV. CONCLUSION

A non-iterative solution for computing the first eccentricity, e of normal ellipsoid is presented. The solution is achieved by considering a series expansion of $\tan^{-1}(x)$ which is used to derive a series approximation solution to the first eccentricity. A sample numerical computation showed that the derived non-iterative solution approach was able to determine the value of the first eccentricity, e of normal ellipsoid with tolerance error of the order of $10^{-9}$. Meanwhile, it took about 10 iterations for Bisection iterative approach to arrive at the solution with tolerance error of the order of $10^{-9}$. As such, the non-iterative solution is reasonably effective and easier in the computation of the first eccentricity, e of the normal ellipsoid.

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