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# Development And Application Of Complementary Root-Based Seeded Secant Iteration For Determination Of Semi Major Axis Of Perturbed Orbit

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***Abstract*— In this paper, development and application of complementary root-based seeded secant (CRSS) iteration method for determination of semi major axis of perturbed orbit is presented. The concept of complementary root is presented along with the detail procedure for the CRSS method and its application in the solution to the orbital equations for the semi major axis and the nominal mean motion of perturbed orbit. A case study perturbed orbit was considered to demonstrate the effectiveness of the CRSS method. According to the result, the initial single guess root value for the semi major axis ($a_0$) is 26,598.53828 km from which a complementary root, g($a_0$) of 26604.74217 km was obtained and the first root (semi major axis, $a_1$ in km) determined using secant method is 26604.7414 km. The specified error tolerance is $1x\,10^{-10}$ km. The results show that it took two (2) cycles for the CRSS to converge at the semi major axis ( $a_2$ ) value of 26604.7414 km with estimation error of $8.36735\,x\,10^{-11}$ km at which point the nominal mean motion ($n_0$) is found to be 0.000145489 rad/s. The result of the case study perturbed orbit clearly shows the effectiveness of the CRSS iteration application in the planetary motion studies.**

***Keywords*— *Seeded Secant, Planetary Motion, Complementary Root, Perturbed Orbit, Numerical Iteration Method, Nominal Mean Motion***

## I. INTRODUCTION

The continual advancements in information and communication technologies have given rise to greater demand for global communication applications and services [1,2,3,4,5,6,7,8] which in turn have led to the deployment of several artificial satellites orbiting the earth [9,10,11,12,13,14,15,16]. Launching and tracking of these satellites in their orbits require accurate knowledge of the orbital parameters [17,18,18,19,20,21,22,23]. The orbits around the oblate earth are significantly affected by the earth's oblateness [24,25,26,27,28,29,30,31,32]. Such orbit around oblate earth are said to be perturbed and the mean motion is a function of the anomalistic period.

However, determination of the nominal mean motion of perturbed orbit from the knowledge of the anomalistic period is quite complex. This is due to the complicated transcendental nature of the expression relating the mean motion and the nominal mean motion of perturbed orbit. As such in this paper a variant of secant iteration method is developed for solving the complex equation for computing the nominal mean motion of perturbed orbit when the anomalistic period is give. The details of the development and application of the complementary root-based seeded secant (CRSS) iteration method is presented along with sample numerical example.

## II DEVELOPMENT OF THE COMPLEMENTARY ROOT–BASED SEEDED SECANT ITERATION

### A. *The Concept of the Complementary Root of Function and the Single Initial Guess Root*

Consider the function of $x$ denoted as $f(x)$ where $f(x)$ is presented in the form;

$$f(x) = x + g(x) = 0$$

where $g(x)$ is another function of x. Hence,

$$x = g(x)$$

Hence, when $x = g(x)$ then, the value of x is the root of the function and it is equal to $g(x)$. When $x \neq g(x)$ the value of x is not the root of $f(x)$. In any case, for any given value of x, the corresponding value of $g(x)$ is the complementary root of x. The more the value of x is closer to the value of $g(x)$, the closer is the value of $f(x)$ to zero (0) . Hence, when $x = g(x) = 0$ , the root of $f(x)$ is found. For instance, consider the function;

$$f(x) = x - x^2 + 2x + 4$$

The function, $f(x)$ can be expressed in the root-complementary root form as;

$$f(x) = x - g(x) = x - (x^2 - 2x - 4)$$

In this case,

$$g(x) = (x^2 - 2x - 4)$$

Then, at x = 1, $g(x) = -5$ and $f(x) = 6$. Also, at x = 3, $g(x) = -1$ and $f(x) = 4$ . However, at x = 4, $g(x) = 4$ and $f(x) = 0$ . Hence, 4 is the root of the function, $f(x) = x - (x^2 - 2x - 4) = x - x^2 + 2x + 4$.

Generally, the complementary root iteration method requires the function, $f(x)$ to be re-presented in the root-complementary root form, $f(x) = x - g(x) = 0$. Then, a single initial guess root, $x_0$ is required to generate the complementary root, $g(x_0)$ . Hence, the two roots

$x_0$ and $g(x_0)$ can be used to carry out secant iteration as follows;

$$x_1 = \frac{(x_0)f(g(x_0)) - (g(x_0))f(x_0)}{f(g(x_0)) - f(x_0)}$$

After the initial roots, $x_0$ and $g(x_0)$ are used to find $x_1$, the secant method is then used in subsequent iterations to find $x_k$ for all K >1. Hence,

$$x_k = \frac{(x_{k-2})f(x_{k-1}) - (x_{k-1})f(x_{k-2})}{f(x_{k-1}) - f(x_{k-2})}$$

In this wise, a seeded secant variant that requires only one initial root guess root value based on the root-complementary root concept has been realized. In this paper, this new variant is referred to as complementary root–based seeded secant (CRSS) iteration method.

### B. *The Procedure for the Complementary Root–Based Seeded Secant Iteration*

In general, the CRSS iteration method can be broken into two parts; part I focuses on the determination of the initial roots and part II focuses on the use of the classical secant iteration to determine the desired root of the function that satisfy the specified tolerance error. The procedure for the CRSS iteration method can be summarized as follows;

**Part I: The initial roots for the secant iterations**

**Step 1.1:**

Express the function in the root-complementary root form as;

$$f(x) = x - g(x) = 0$$

**Step 1.2:**

Input the initial single guess root value, $x_0$ and the tolerance error , ∈

**Step 1.3:**

Compute the complementary root $g(x_0)$

**Step 1.4:**

Use the secant method to generate the second root , $x_1$ as follows;

$$x_1 = \frac{(x_0)f(g(x_0)) - (g(x_0))f(x_0)}{f(g(x_0)) - f(x_0)}$$

**Part II: The classical secant iterations**

**Step 2.1:**

$$x_k = x_0$$

**Step 2.2:**

$$x_{k+1} = x_1$$

**Step 2.3:**

$$k = 2$$

**Step 2.3:**

$$x_k = \frac{(x_{k-2})f(x_{k-1}) - (x_{k-1})f(x_{k-2})}{f(x_{k-1}) - f(x_{k-2})}$$

**Step 2.5:**

If $f(x_k) > \in$ then

**Step 2.5.1.1:**

k =k +1

**Step 2.5.1.2:**

Goto **Step 2.4:**

Else

**Step 2.5.2.1:**

Output $x_k$ , k

Endif

## III. APPLICATION OF COMPLEMENTARY ROOT–BASED SEEDED SECANT ITERATION IN PERTURBED ORBIT PARAMETER COMPUTATION

### A. *Analytical Expression for Computing the Semi Major Axis and the Nominal Mean Motion of Perturbed Orbit*

Notably, in this paper, value of the semi major axis (a) and the nominal mean motion $(n_o)$ are determined based on a given anomalistic period of the orbit. When the anomalistic period (P) is given, the orbit mean motion, denoted as n, is given as;

The semi major axis (a) and the nominal mean motion $(n_o)$ of an orbit that is not perturbed are related as follows;

$$n_o = \sqrt{\frac{\mu}{a^3}} \quad (1)$$

Similarly, the perturbed orbit mean motion (n) is related to the orbit anomalistic period (p) as follows;

$$n = \frac{2\pi}{P} \quad (2)$$

Furthermore, the perturbed orbit mean motion (n) is related to the nominal mean motion $(n_o)$ and the semi major axis (a) given as follows;

$$n = \frac{2\pi}{P} = n_o\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{a^2(1-e^2)^{1.5}}\right] = \sqrt{\frac{\mu}{a^3}}\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{a^2(1-e^2)^{1.5}}\right] \quad (3)$$

### B. *The Root-Complementary Root Form of the Function for Semi Major Axis (a)*

In this paper, the semi major axis (a) is first determined when the anomalistic period of the perturbed orbit is provided. In order to apply the complementary root–based seeded secant iteration for he determination of the semi major axis, the function of the semi major axis must be re-presented in the root-complementary root form. From Eq 3, the semi major axis (a) is given as follows;

$$\frac{n^2}{\left[1+\frac{K_1(1-1.5\sin(i)^2)}{a^2(1-e^2)^{1.5}}\right]^2} = \frac{\mu}{a^3} \quad (4)$$

Hence;

$$a = \left(\frac{\mu}{n^2}\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{a^2(1-e^2)^{1.5}}\right]^2\right)^{1/3} = \left(\frac{\mu}{\left(\frac{2\pi}{P}\right)^2}\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{a^2(1-e^2)^{1.5}}\right]^2\right)^{1/3} \quad (5)$$

The root-complementary root form

$$f(a_k) = a_k - \left(\frac{\mu}{n^2}\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{(a_k)^2(1-e^2)^{1.5}}\right]^2\right)^{1/3} = a_k - \left(\frac{\mu}{\left(\frac{2\pi}{P}\right)^2}\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{(a_k)^2(1-e^2)^{1.5}}\right]^2\right)^{1/3} \quad (6)$$

Essentially, the complementary root function, $g(a_k)$ is given as

$$g(a_k) = \left(\frac{\mu}{n^2}\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{(a_k)^2(1-e^2)^{1.5}}\right]^2\right)^{1/3} = \left(\frac{\mu}{\left(\frac{2\pi}{P}\right)^2}\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{(a_k)^2(1-e^2)^{1.5}}\right]^2\right)^{1/3} \quad (7)$$

### *C. Application of the CRSS Iteration Procedure for the Computation of the Semi Major Axis of a Perturbed Orbit*

The CRSS iteration procedure adapted for the computation of the semi major axis (a) of a perturbed orbit can be summarized as follows;

**Part I: The initial roots for the secant iterations**

**Step 1.1:**

Express the function in the root-complementary root form as;

$$f(a_k) = a_k - \left(\frac{\mu}{n^2}\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{(a_k)^2(1-e^2)^{1.5}}\right]^2\right)^{1/3} = a_k - \left(\frac{\mu}{\left(\frac{2\pi}{P}\right)^2}\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{(a_k)^2(1-e^2)^{1.5}}\right]^2\right)^{1/3}$$

**Step 1.2:**

Input the initial single guess root value, $a_0$ and the tolerance error , ∈

**Step 1.3:**

Compute the complementary root $g(a_0)$ where

$$g(a_0) = \left(\frac{\mu}{n^2}\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{(a_0)^2(1-e^2)^{1.5}}\right]^2\right)^{1/3} = \left(\frac{\mu}{\left(\frac{2\pi}{P}\right)^2}\left[1 + \frac{K_1(1-1.5\sin(i)^2)}{(a_0)^2(1-e^2)^{1.5}}\right]^2\right)^{1/3}$$

**Step 1.4:**

Use the secant method to generate the second root , $x_1$ as follows;

$$a_1 = \frac{(a_0)f(g(a_0)) - (g(a_0))f(a_0)}{f(g(a_0)) - f(a_0)}$$

**Part II: The classical secant iterations**

**Step 2.1:**

$$a_k = a_0$$

**Step 2.2:**

$$a_{k+1} = a_1$$

**Step 2.3:**

$$k = 2$$

**Step 2.3:**

$$a_k = \frac{(a_{k-2})f(a_{k-1}) - (a_{k-1})f(a_{k-2})}{f(a_{k-1}) - f(a_{k-2})}$$

**Step 2.5:**

If $f(a_k) > \in$ then

**Step 2.5.1.1:**

k =k +1

**Step 2.5.1.2:**

Goto **Step 2.4:**

Else

**Step 2.5.2.1:**

Output $a_k$ , k

Endif

## IV. RESULTS AND DISCUSSION

The complementary root-based seeded secant (CRSS) iteration was applied in the computation of the semi major axis as well as the nominal mean motion of a case study perturbed orbit having the orbital parameters shown in Table 1. The results of the initial value determination and the secant iteration for the case study perturbed orbit are shown in Table 2. According to the result, the initial single guess root value for the semi major axis ( $a_0$ ) is 26,598.53828 km from which a complementary root, $g(a_0)$ of 26604.74217 km was obtained and the first root (semi major axis, $a_1$ in km) determined using secant method is 26604.7414 km. The specified error tolerance is 1x $10^{-10}$ km. As such, the results in Table 2 show that it took about two (2) cycles for the CRSS to converge at the semi major axis ($a_3$) value of 26604.7414km with estimation error of 8.36735 x $10^{-11}$ km at which point the nominal mean motion ($n_0$) is found to be 0.000145489 rad/s. The result of the case study perturbed orbit clearly shows the effectiveness of the CRSS iteration application in the planetary motion studies.

**Table 1 The case study perturbed orbit parameters**

| S/N | PARAMETER NAME AND SYMBOL | PARAMETER VALUE AND UNIT |
|---|---|---|
| 1 | Anomalistic Period (P) | 11.98 hour |
| 2 | Inclination Angle (i) | 0 degree |
| 3 | Eccentricity (e) | 0.0018 |
| 4 | Constant ($K_1$) | 66,063.1704 $km^2$ |
| 5 | Earth Geocentric Gravitational Constant ($\mu$) | 3.986005 x$10^{14}$ $m^3/s^2$ |

**Table 2 The results of the initial value determination and the secant iteration for the case study perturbed orbit**

| Result of the initial root determination for the secant iteration for K = 0 | | | |
|---|---|---|---|
| Cycle | Initial guess semi major axis, $a_0$ in km | Complementary root, $g(a_0)$ of the initial guess semi major axis, $a_0$ in km | First root (semi major axis, $a_1$ in km) determined using secant method |
| 0 | 26,598.53828 | 26604.74217 | 26604.7414 |

| Result of the secant iteration for K > 1 | | | | | |
|---|---|---|---|---|---|
| Cycle | Semi major axis, $a_k$ in km | Complementary root, $g(a_k)$ in km | Error , $f(a_k)$ in km | Nominal mean motion $(n_0)$ in rad/s | Mean motion (n) in rad/s |
| 1 | 26604.74217 | 26604.7414 | 0.00077225 | 0.000145489 | 0.000145503 |
| 2 | 26604.7414 | 26604.7414 | 8.36735E-11 | 0.000145489 | 0.000145503 |
| 3 | 26604.7414 | 26604.7414 | 0 | 0.000145489 | 0.000145503 |
| 4 | 26604.7414 | 26604.7414 | 0 | 0.000145489 | 0.000145503 |

## V. CONCLUSION

A variant of scant iteration method is developed and applied for computing the semi major axis of a perturbed orbit as well as its nominal mean motion. The new scant iteration variant is called complementary root-based seeded secant (CRSS) iteration method. The concept of complementary root is presented along with the detail procedure for the CRSS method and its application in the solution to the equations for the semi major axis of a perturbed orbit as well as its nominal mean motion. A case study perturbed orbit was considered to demonstrate the effectiveness of the CRSS method.

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